

MAS317 Linear Algebra II, 2007
Solutions to Coursework 2

1. $q(x_1, x_2, x_3) = x_1^2 + 6x_1x_2 - 4x_1x_3 + 6x_2x_3 + 7x_3^2$

To find y_1 , take all the terms involving x_1 , and note that

$$x_1^2 + 6x_1x_2 - 4x_1x_3 = (x_1 + 3x_2 - 2x_3)^2 - 9x_2^2 + 12x_2x_3 - 4x_3^2,$$

so putting $y_1 = x_1 + 3x_2 - 2x_3$ and substituting into q and collecting terms we have: $q(x_1, x_2, x_3) = y_1^2 - 9x_2^2 + 18x_2x_3 + 3x_3^2$. **[8 marks]**

To find y_2 , take all the terms in this expression involving x_2 , and note that

$$-9x_2^2 + 18x_2x_3 = -9(x_2^2 - 2x_2x_3) = -9(x_2 - x_3)^2 + 9x_3^2$$

so putting $y_2 = x_2 - x_3$ and substituting into q and collecting terms we have: $q(x_1, x_2, x_3) = y_1^2 - 9y_2^2 + 12x_3^2$. **[4 marks]**

So, if $y_1 = x_1 + 3x_2 - 2x_3$, $y_2 = x_2 - x_3$, $y_3 = x_3$, then

$$q(x_1, x_2, x_3) = y_1^2 - 9y_2^2 + 12y_3^2 \text{ and so } D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & 12 \end{pmatrix} \quad \mathbf{[2 \text{ marks}]}$$

Now $x_3 = y_3$, $x_2 = y_2 + y_3$, $x_1 = y_1 - 3y_2 - y_3$

$$\text{so } P^TAP = D \text{ if } P = \begin{pmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \quad \mathbf{[6 \text{ marks}]}$$

2. Define $q(x_1, x_2, x_3) = 4x_1x_2 + 4x_1x_3 - 2x_2x_3$. This has matrix B .

There are no squares in this. One way to introduce some is to notice that $(x_1 + x_2)^2 - (x_1 - x_2)^2 = 4x_1x_2$, so putting $z_1 = x_1 + x_2$ and $z_2 = x_1 - x_2$ and $z_3 = x_3$, we get $q(x_1, x_2, x_3) = q'(z_1, z_2, z_3)$, where

$$q'(z_1, z_2, z_3) = z_1^2 - z_2^2 + 4x_1z_3 - 2x_2z_3$$

$$= z_1^2 - z_2^2 + 2(z_1 + z_2)z_3 - (z_1 - z_2)z_3$$

$$= z_1^2 + z_1z_3 - z_2^2 + 3z_2z_3.$$

[10 marks]

Now as in question 1, use the terms involving z_1 to find y_1 etc....

$$z_1^2 + z_1z_3 = (z_1 + (1/2)z_3)^2 - (1/4)z_3^2, \text{ so put } y_1 = z_1 + (1/2)z_3.$$

[4 marks]

Now

$$q'(z_1, z_2, z_3) = y_1^2 - (z_2^2 - 3z_2z_3) - (1/4)z_3^2.$$

$$= y_1^2 - y_2^2 + 2z_3^2, \text{ if } y_2 = z_2 - (3/2)z_3. \quad [4 \text{ marks}]$$

So put $y_3 = z_3$, then $q(y_1, y_2, y_3) = y_1^2 - y_2^2 + 2y_3^2$ and so

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad [2 \text{ marks}]$$

Now $y_1 = z_1 + (1/2)z_3 = x_1 + x_2 + (1/2)x_3$ and

$y_2 = z_2 - (3/2)z_3 = x_1 - x_2 - (3/2)z_3$ and $y_3 = z_3 = x_3$,

and so $x_3 = y_3$, and $2x_2 = y_1 - y_2 - 2y_3$, and $2x_1 = y_1 + y_2 + y_3$ so if

$$P = \begin{pmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1 \\ 0 & 0 & 1 \end{pmatrix} \text{ then } P^T B P = D. \quad [10 \text{ marks}]$$

3. $f((x_1, x_2, x_3), (y_1, y_2, y_3)) =$
 $x_1y_1 + 3x_1y_2 - 2x_1y_3 + 3x_2y_1 + 3x_2y_3 - 2x_3y_1 + 3x_3y_2 + 7x_3y_3.$

First find v_1 s.t. $f(v_1, v_1) \neq 0$. *There is lots of choice here.*

Eg take $v_1 = (1, 0, 0)$ as $f((1, 0, 0), (1, 0, 0)) = 1$. [5 marks]

Now v_2 and v_3 must be orthogonal to v_1 wrt f . (x_1, x_2, x_3) is
orthogonal to v_1 wrt f if $f((x_1, x_2, x_3), (1, 0, 0)) = 0$ and so
 $x_1 + 3x_2 - 2x_3 = 0$.

So v_2 must be a solution to this and have $f(v_2, v_2) \neq 0$; do this by
trial and error.

Eg take $v_2 = (-3, 1, 0)$, then

$$f((x_1, x_2, x_3), v_2) = -3x_1 + 3x_1 - 9x_2 + 6x_3 + 3x_3 = -9x_2 + 9x_3. \text{ So } f(v_2, v_2) = -9 \text{ (non-zero as required).} \quad [8 \text{ marks}]$$

Now v_3 must be orthogonal to v_1 and to v_2 , so if $v_3 = (x_1, x_2, x_3)$ then
 $x_1 + 3x_2 - 2x_3 = 0$ and $-9x_2 + 9x_3 = 0$.

Eg take $v_3 = (-1, 1, 1)$. Then

$$f(v_3, v_3) = 1 - 3 + 2 - 3 + 3 + 2 + 3 + 7 = 12. \quad [8 \text{ marks}]$$

$$\text{Now } D = \begin{pmatrix} f(v_1, v_1) & 0 & 0 \\ 0 & f(v_2, v_2) & 0 \\ 0 & 0 & f(v_3, v_3) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & 12 \end{pmatrix}, \quad [2 \text{ marks}]$$

and P has the vectors v_1, v_2, v_3 as its columns so if

$$P = \begin{pmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{ then } P^T A P = D. \quad [2 \text{ marks}]$$

4. $B = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & -1 \\ 2 & -1 & 0 \end{pmatrix}$. Define the bilinear form g by [3 marks]

$$g((x_1, x_2, x_3), (y_1, y_2, y_3)) = 2x_1y_2 + 2x_1y_3 + 2x_2y_1 - x_2y_3 + 2x_3y_1 - x_3y_2.$$

First find v_1 s.t. $g(v_1, v_1) \neq 0$. *There is lots of choice here, but $(1, 0, 0)$ will not do;*

Eg take $v_1 = (1, 1, 0)$, then $g((x_1, x_2, x_3), v_1) = 2x_1 + 2x_2 + x_3$, and so $g(v_1, v_1) = 4$. [6 marks]

Now v_2 and v_3 must be orthogonal to v_1 wrt g .

(x_1, x_2, x_3) is orthogonal to v_1 wrt g if $2x_1 + 2x_2 + x_3 = 0$.

So v_2 must be a solution to this and have $g(v_2, v_2) \neq 0$; do this by trial and error, choose a vector and find $g(v_2, v_2)$. *There is lots of choice here too;*

Eg take $v_2 = (1, -1, 0)$, then $g((x_1, x_2, x_3), v_2) = -2x_1 + 2x_2 + 3x_3$. So $g(v_2, v_2) = -4$ (non-zero as required). [6 marks]

Now v_3 must be orthogonal to v_1 and to v_2 , so if $v_3 = (x_1, x_2, x_3)$ then $2x_1 + 2x_2 + x_3 = 0$ and $-2x_1 + 2x_2 + 3x_3 = 0$.

Eg take $v_3 = (1, -2, 2)$. Then $g(v_3, v_3) = -4 + 4 - 4 + 4 + 4 = 4 = 8$.

[6 marks]

$$\text{Now } D = \begin{pmatrix} g(v_1, v_1) & 0 & 0 \\ 0 & g(v_2, v_2) & 0 \\ 0 & 0 & g(v_3, v_3) \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 8 \end{pmatrix}, \quad [2 \text{ marks}]$$

and P has the vectors v_1, v_2, v_3 as its columns so if

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -2 \\ 0 & 0 & 2 \end{pmatrix} \text{ then } P^T A P = D. \quad [2 \text{ marks}]$$